Topological rigidity of automorphism actions on nilmanifolds

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1 Introduction

Let G be connected simply connected nilpotent Lie group and D be discrete uniform subgroup of G. Then $X = G/D$ is called a nilmanifold. If $X_1 = G_1/D_1$ and $X_2 = G_2/D_2$ are nilmanifolds then a map $f : X_1 \to X_2$ is said to be a *homomorphism* if it is induced by a continuous homomorphism from G_1 to G_2 which maps D_1 into D_2. Isomorphisms and automorphisms are defined similarly. A map $f : X_1 \to X_2$ is said to be *affine* if there exists an element a in G_2 and a continuous homomorphism $A : G_1 \to G_2$ such that $f(gD_1) = aA(g)D_2$ for all g in G_1. If Γ is a discrete group and X is a nilmanifold then a Γ-action ρ on X is said to be an *automorphism action* if each $\rho(\gamma)$ is an automorphism of X.

It is known that any nilmanifold $X = G/D$ carries a unique G-invariant probability measure λ_X. It is easy to see that for any nilmanifold X, the measure λ_X is invariant under any affine action on X. An automorphism action ρ of a discrete group Γ on a nilmanifold X is said to be *ergodic* if for every Γ-invariant function f in $L^2(X, \lambda_X)$ is a constant almost everywhere.

Suppose X_1, X_2 are nilmanifolds and ρ, σ are continuous actions of a discrete group Γ on X_1 and X_2 respectively. If $f : X_1 \to X_2$ is a continuous

map and Γ_0 is a subgroup of Γ then f is said to be Γ_0-*equivariant* if $f \circ \rho(\gamma) = \sigma(\gamma) \circ f, \quad \forall \gamma \in \Gamma_0$. A continuous map $f : X_1 \to X_2$ is said to be *almost equivariant* if there exists a finite-index subgroup $\Gamma_0 \subset \Gamma$ such that f is Γ_0-equivariant.

In this paper we show that if ρ, σ are automorphism actions on nilmanifolds satisfying certain conditions then every Γ-equivariant continuous map from (X_1, ρ) to (X_2, σ) is an affine map. When $\Gamma = \mathbb{Z}$, (X_2, σ) is a factor of (X_1, ρ) and ρ, σ are generated by affine transformations, this phenomenon has been studied in [AP],[Wa1] and [Wa2]. In this case a necessary and sufficient condition for existence of a non-affine Γ-equivariant map is given in [Wa2]. Our methods are however different and applicable in more general situations.

This paper is organized as follows. In section 2 we study structure of continuous equivariant maps from (X_1, ρ) to (X_2, σ). In Theorem 1 we give a necessary and sufficient condition for existence of a non-affine almost equivariant map. For any Lie group G, by $L(G)$ we denote the Lie algebra of G. If ρ is an automorphism action of a discrete group Γ on a nilmanifold $X = G/D$, then by ρ_e we denote the Γ-action on $L(G)$ induced by ρ. We prove the following.

Theorem 1 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and ρ, σ be automorphism actions of a discrete group Γ on X_1 and X_2 respectively. Then there exists a non-affine almost equivariant continuous map from (X_1, ρ) to (X_2, σ) if and only if the following two conditions are satisfied.*

a) There exists a non-constant Γ-invariant continuous function from (X_1, ρ) to $\mathbb{R}$.

b) There exists a nonzero vector v in $L(G_2)$ with finite σ_e-orbit.

As a consequence we obtain that if either (X_1, ρ) is ergodic or (X_2, σ) is expansive then every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map (see corollary 2.1).

In section 3 we consider the case when X_1 is a torus. If ρ is an automorphism action of a discrete group Γ on a torus T^m then we denote the induced automorphism action of Γ on the dual group $\widehat{T^m}$ by $\widehat{\rho}$. By F_ρ we denote the subgroup of $\widehat{T^m}$ which consists of all elements with finite $\widehat{\rho}$-orbit and by Γ_ρ we denote the subgroup of Γ consisting of all elements which acts trivially on F_ρ under the action $\widehat{\rho}$. We prove the following.

Theorem 2 : *Let Γ be a discrete group, T^m be the m-torus and $X = G/D$ be a nilmanifold. Let ρ, σ be automorphism actions of Γ on T^m and X respectively. Then there exists a non-affine continuous Γ-equivariant map from (T^m, ρ) to (X, σ) if and only if the following two conditions are satisfied.*

- *a) (T^m, ρ) is not ergodic.*
- *b) There exists a nonzero vector v in $L(G)$ which is fixed by Γ_ρ under the action σ_e.*

In section 4 we consider the case when Γ is abelian and (X_2, σ) is a topological factor of (X_1, ρ) i.e. there exists a continuous Γ-equivariant map from (X_1, ρ) onto (X_2, σ). Generalizing the corresponding results in [Wa1] and [Wa2] we obtain the following.

Theorem 3 : *Let X_1, X_2 be nilmanifolds and ρ, σ be automorphism ac-*

tions of a discrete abelian group Γ on X_1 and X_2 respectively. Suppose that (X_2, σ) is a factor of (X_1, ρ) and either $X_1 = X_2$ or X_2 is a torus. Then there is a non-affine continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) if and only if (X_2, σ) is not ergodic.

2 Almost equivariant maps

In this section we give a necessary and sufficient condition for existence of a non-affine almost equivariant map from (X_1, ρ) to (X_2, σ). Throughout this section for $i = 1, 2$; $X_i = G_i/D_i$ will denote a nilmanifold, π_i will denote the projection map from G_i to X_i, e_i will denote the identity element of G_i and $\bar{e}_i$ will denote the image of e_i in X_i under the map π_i. It is known that in this case any homomorphism from D_1 to D_2 can be extended to a continuous homomorphism from G_1 to G_2 (cf. [Ma]). We will also use the following fact (cf. [AGH], pp. 54).

Proposition 2.1 ([AGH]) **:** *Let $X = G/D$ be a nilmanifold. For $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ let I_a denote the set defined by*

$$I_a = \{ \ x \mid a_i \leq x_i \leq a_i + 1 \ \forall i = 1, \ldots, n \ \}.$$

Then there exists an invertible linear map T from $\mathbb{R}^n$ to $L(G)$ such that for all a in $\mathbb{R}^n$, the set $\exp \circ T(I_a)$ is a fundamental domain for $X = G/D$.

The following proposition was proved in [Wa2]. (see also [AP]).

Proposition 2.2 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and $f : X_1 \to X_2$ be a continuous map. Let $F : G_1 \to G_2$ be a lift of f. Then*

there exist a $g_0 \in G_2$, a continuous homomorphism $\theta(f) : G_1 \to G_2$ and a continuous map $P(f) : G_1 \to G_2$ such that

a) $P(f)(e_1) = e_2,\ P(f)(g \cdot \gamma) = P(f)(g) \ \ \forall g \in \Gamma.$

b) $F(g) = P(f)(g) \cdot g_0 \cdot \theta(f)(g) \ \ \forall g \in \Gamma.$

Moreover for a given f, the maps $\theta(f)$ and $P(f)$ are unique.

Suppose $X_1,\ X_2$ are nilmanifolds and $\rho,\ \sigma$ are automorphism actions of a discrete group Γ on X_1 and X_2 respectively. Let $\overline{\rho}$ and $\overline{\sigma}$ denote the induced automorphism actions of Γ on G_1 and G_2 respectively. Then from the uniqueness part of Proposition 2.2 it is follows that for any Γ-equivariant continuous map f from (X_1, ρ) to (X_2, σ), $P(f)$ is a Γ-equivariant continuous map from $(G_1, \overline{\rho})$ to $(G_2, \overline{\sigma})$. Now we obtain the following.

Lemma 2.1 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and $\rho,\ \sigma$ be automorphism actions of a discrete group Γ on X_1 and X_2 respectively. Then there exists a nonaffine continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) if and only if there exists a nonzero continuous Γ-equivariant map S from (X_1, ρ) to $(L(G_2), \sigma_e)$ such that $S(\bar{e_1}) = 0$.*

Proof : Suppose there exists a nonaffine Γ-equivariant continuous map f from (X_1, ρ) to (X_2, σ). Let $P = P(f) : G_1 \to G_2$ be as defined above. Since $P(e_1) = e_2$ and $P(g \cdot \gamma) = P(g)$ for all $g \in \Gamma$, there exists a unique continuous map Q from X_1 to G_2 such that $Q(\overline{e_1}) = e_2$ and $P = Q \circ \pi_1$. It is easy to see that Q is a Γ-equivariant map from (X_1, ρ) to $(G_2, \overline{\sigma})$. Note that since G_2 is a connected simply connected nilpotent Lie group, the map $\exp : L(G_2) \to G_2$ is a diffeomorphism. Hence there is a unique map $S : X_1 \to L(G_2)$ such that $Q = \exp \circ S$. Now $S(\overline{e_1}) = 0$ and since exp is a Γ-equivariant map from

$(L(G_2), \sigma_e)$ to $(G_2, \overline{\sigma})$ it is easy to see that S is a Γ-equivariant map from (X_1, ρ) to $(L(G_2), \sigma_e)$. Since f is a nonaffine map, $P(f) = \exp \circ S \circ \pi_1$ is non-constant i.e. S is a nonzero map.

Now suppose there exists a non-zero Γ-equivariant continuous map S from (X_1, ρ) to $(L(G_2), \sigma_e)$ such that $S(\bar{e_1}) = 0$. Define a map $f : X_1 \to X_2$ by

$$f(x) = \pi_2 \circ \exp \circ S(x) \ \forall x \in X_1.$$

It is easy to check that f is a Γ-equivariant map from (X_1, ρ) to (X_2, σ) and $P(f) = \exp \circ S \circ \pi_1$. Since the map $\exp \circ S$ is non-constant, so is P. Now from the uniqueness part of Proposition 2.2 it follows that f is a nonaffine map.

Lemma 2.2 : *Let $X = G/D$ be a nilmanifold and V be a finite dimensional vector space over $\mathbb{R}$. Let Γ be a discrete group and ρ, σ be automorphism actions of Γ on X and V respectively. Then for any Γ-equivariant map $S : X \to V$ there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that $S \circ \rho(\gamma) = S \quad \forall \gamma \in \Gamma_0$.*

Proof : We define $A, A_1, A_2, \ldots \subset L(G)$ by

$$A_i = \{v \mid \exp(iv) \in D\} \ , \quad A = \cup A_i.$$

If $\pi : G \to G/D$ denotes the projection map then we define $B, B_1, B_2, \ldots \subset X$ by

$$B_i = \pi \circ \exp(A_i) \ , \quad B = \cup B_i.$$

From Proposition 2.1 it follows that each B_i is a finite subset of X and B is dense in X. Also it is easy to see that each B_i is invariant under the action ρ. Therefore for any element x in B, the ρ-orbit of x is finite. Let W denote

the subspace of V which consists of all elements of V whose σ-orbit is finite. Since S is Γ-equivariant and B is a dense subset of X, it follows that the image of S is contained in W. Now choose a basis $\{w_1, w_2, \ldots w_l\}$ of W. Define $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ and Γ_0 by

$$\Gamma_i = \{\gamma \in \Gamma \mid \sigma(\gamma)(w_i) = w_i \ \}, \quad \Gamma_0 = \cap \Gamma_i.$$

Since each $\Gamma_i \subset \Gamma$ is a subgroup of finite index, so is Γ_0. Since Γ_0 acts trivially on W and image of S is contained in W, we conclude that S is a Γ_0-invariant map.

Proof of Theorem 1 : Suppose there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and a nonaffine continuous map f from (X_1, ρ) to (X_2, σ) which is Γ_0-equivariant. Then by Lemma 2.1 there exists a nonzero continuous Γ_0-equivariant map S from (X_1, ρ) to $(L(G_2), \sigma_e)$ such that $S(\bar{e_1}) = 0$. Let W denote the subspace of $L(G_2)$ which consists of all elements of $L(G_2)$ whose σ_e-orbit is finite. Then from Lemma 2.2 it follows that the image of S is contained in W. Hence there exists a nonzero vector v in $L(G_2)$ such that the σ_e-orbit of v is finite. To prove a) we choose a norm $||.||$ on W and define a function $p : W \mapsto \mathbb{R}$ by

$$p(w) = \text{Inf } \{||\sigma_e(\gamma)(w)|| \mid \gamma \in \Gamma \ \}.$$

Since σ_e-orbit of any element of W is finite, the map $q = p \circ S$ is a non-constant continuous Γ-invariant function from X_1 to $\mathbb{R}$.

Now suppose both the conditions a) and b) are satisfied. Then W is a nonzero subspace of $L(G_2)$ and there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that the σ_e-action of Γ_0 on W is trivial. Let $q : X_1 \mapsto \mathbb{R}$ be a non-constant continuous Γ-invariant function from X_1 to $\mathbb{R}$ and let $h : \mathbb{R} \mapsto W$

be a continuous map such that the map $h \circ q$ is nonzero and $h \circ q(\bar{e_1}) = 0$. Then $S = h \circ q$ is a nonzero continuous Γ_0-equivariant map from (X_1, ρ) to $(L(G_2), \sigma_e)$ and $S(\bar{e_1}) = 0$. Applying Lemma 2.1 we see that there exists a nonaffine continuous Γ_0-equivariant map from (X_1, ρ) to (X_2, σ).

Let (X, d) be a metric space and ρ be a continuous action of a group Γ on X. Then (X, ρ) is said to be *expansive* if there exists $\epsilon > 0$ such that for any two distinct points x, y in X,

$$\text{Sup } \{ \ d(\rho(\gamma)(x), \rho(\gamma)(y)) \mid \gamma \in \Gamma \ \} \geq \epsilon.$$

Any such ϵ is called an expansive constant of (X, ρ). It is easy to check that the notion of expansiveness is independent of the metric d.

Now as a corollary of Theorem 1 we obtain the following.

Corollary 2.1 : *Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and ρ, σ be automorphism actions of a discrete group Γ on X_1 and X_2 respectively. Suppose that either (X_1, ρ) is ergodic or (X_2, σ) is expansive. Then every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map.*

Proof : If (X_1, ρ) is ergodic then there is no non-constant Γ-invariant continuous function from (X_1, ρ) to $\mathbb{R}$. Applying Theorem 1 we see that there exists no nonaffine continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) .

Suppose that (X_2, σ) is expansive. Choose a metric d on X_2 and an expansive constant $\epsilon > 0$ with respect to d. Define open sets $U \subset X_2$ and $V \subset L(G_2)$ by

$$U = \{x \mid d(\bar{e}_2, x) < \epsilon \ \} \ , \quad V = (\pi_2 \circ \exp)^{-1}(U).$$

We claim that for every nonzero vector v in $L(G_2)$, the σ_e-orbit of v is infinite. To see this choose any vector v_0 in $L(G_2)$ such that the σ_e-orbit of v_0 is finite. Choose $\alpha > 0$ sufficiently small so that the σ_e-orbit of αv_0 is contained in V and does not intersect the set $\exp^{-1}(D_2) - \{0\}$. Then the σ-orbit of the element $x_0 = \pi_2 \circ \exp(\alpha v_0)$ is contained in U. Since $\bar{e_2}$ is fixed by the action σ, it follows that $x_0 = \bar{e_2}$ i.e. $v_0 = 0$. Now applying Theorem 1 we see that every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map.

3 Rigidity of toral automorphisms

In this section we will consider automorphism actions of discrete groups on tori. Suppose ρ is an automorphism action of a discrete group Γ on T^m. Then $\widehat{\rho}$ will denote the automorphism action of Γ on T^m defined by

$$\widehat{\rho}(\gamma)(\chi) = \chi \circ \rho(\gamma) \quad \forall \chi \in \widehat{T}^m, \gamma \in \Gamma.$$

It is well known that (T^m, ρ) is ergodic if and only if $\widehat{\rho}$ has no nontrivial finite orbit. Recall that if (T^m, ρ) is not ergodic then $F_\rho \subset \widehat{T}^m$ will denote the subgroup consisting of all elements with finite $\widehat{\rho}$-orbit and $\Gamma_\rho \subset \Gamma$ will denote the subgroup defined by

$$\Gamma_\rho = \{\gamma \mid \chi \circ \rho(\gamma) = \chi \ \forall \chi \in F_\rho \ \}.$$

Since F_ρ is a finitely generated group, it follows that $\Gamma_\rho \subset \Gamma$ is a subgroup of finite index.

Lemma 3.1 : *Suppose Γ, ρ and Γ_ρ are as above. Then there exists $\chi_0 \in \widehat{T}^m$ and $x_0 \in T^m$ such that*

$$\Gamma_\rho \ = \ \{\gamma \mid \chi_0 \circ \rho(\gamma)(x_0) = \chi_0(x_0)\}$$

Proof : For any χ in F_ρ, let $\Gamma_\chi \subset \Gamma$ denote the stabilizer of χ under the Γ-action $\widehat{\rho}$. We claim that for any χ_1, χ_2 in F_ρ, there exists a $\chi^{'}$ in F_ρ such that $\Gamma_{\chi^{'}} = \Gamma_{\chi_1} \cap \Gamma_{\chi_2}$. To see this, for $i = 1, 2$ define $A_i \subset \widehat{T}^m$ by

$$A_i \;=\; \{\chi_i \circ \rho(\gamma) - \chi_i \mid \gamma \in \Gamma\;\}.$$

Since χ_1, χ_2 are elements of F_ρ, both A_1 and A_2 are finite. Choose n large enough so that $nA_1 \cap A_2 = \{0\}$. Define $\chi^{'} = n\chi_1 - \chi_2$. Clearly $\Gamma_{\chi_1} \cap \Gamma_{\chi_2}$ is contained in $\Gamma_{\chi^{'}}$. On the other hand if $\gamma \in \Gamma_{\chi^{'}}$ then

$$n(\chi_1 \circ \rho(\gamma) - \chi_1) = \chi_2 \circ \rho(\gamma) - \chi_2.$$

Since $nA_1 \cap A_2 = \{0\}$, this implies that $\gamma \in \Gamma_{\chi_1} \cap \Gamma_{\chi_2}$.

Suppose $\chi_1, \ldots, \chi_d$ is a finite set of generators of F_ρ. From the above claim it follows that that there exists a χ_0 in F_ρ such that

$$\Gamma_{\chi_0} = \Gamma_{\chi_1} \cap \cdots \cap \Gamma_{\chi_d} = \Gamma_\rho.$$

Let $x_0 \in T^m$ be any element such that the cyclic subgroup generated by x_0 is dense in T^m. Then it is easy to see that

$$\Gamma_\rho \;=\; \Gamma_{\chi_0} \;=\; \{\gamma \mid \chi_0 \circ \rho(\gamma)(x_0) = \chi_0(x_0)\}.$$

Lemma 3.2 : *Let $\Gamma_1 \subset \Gamma$ be a subgroup of finite index and f be a Γ_1-invariant continuous map from (T^m, ρ) to a metric space (Y, d). Then f is Γ_ρ-invariant.*

Proof : First let us assume that $(Y, d) = \mathbb{C}$ with the usual metric. Let

$\widehat{f} : \widehat{T}^m \to \mathbb{C}$ be the Fourier transform of f. It is easy to check that f is invariant under a subgroup $\Gamma_2 \subset \Gamma$ if and only if $\widehat{f}$ is constant on each Γ_2-orbit under the action $\widehat{\rho}$. Since Γ_ρ acts trivially on F_ρ under the action $\widehat{\rho}$, to prove Γ_ρ-invariance of f it is sufficient to show that $\widehat{f} = 0$ on $\widehat{T}^m - F_\rho$.

Since Γ_1 is a subgroup of finite index, for any ϕ in $\widehat{T}^m - F_\rho$, the Γ_1-orbit of ϕ is infinite. Since f is Γ_1-invariant, $\widehat{f}$ is constant on the Γ_1-orbit of ϕ. Since $\sum_\phi |\widehat{f}(\phi)|^2 < \infty$, we conclude that $\widehat{f}(\phi) = 0$.

Now let (Y, d) be any arbitrary metric space and f be a continuous Γ_1-invariant function from T^m to Y. If $C(Y, \mathbb{C})$ denotes the set of all continuous functions from Y to $\mathbb{C}$, then for each g in $C(Y, \mathbb{C})$ the map $g \circ f$ is Γ_1-invariant. Since $C(Y, \mathbb{C})$ separates points of Y, from the previous argument it follows that f is Γ_ρ-invariant.

Proof of Theorem 2 : Suppose there exists a non-zero continuous Γ-equivariant map from (T^m, ρ) to (X, σ). Then the condition a) follows from Corollary 2.1. Also from Lemma 2.1 it follows that there exists a non-zero continuous Γ-equivariant map S from (T^m, ρ) to $(L(G), \sigma_e)$. Applying Lemma 2.2 and Lemma 3.2 we see that S is Γ_ρ-invariant. This implies that the σ_e-action of Γ_ρ on the image of S is trivial. Now the condition b) follows from the fact that S is nonzero.

Now suppose the conditions a) and b) are satisfied. Fix a finite subset $A = \{\gamma_1, \ldots, \gamma_d\}$ of Γ which contains exactly one element of each right coset of Γ_ρ. Let W denote the subspace of $L(G)$ which is fixed by $\sigma_e(\gamma)$ for all γ in Γ_ρ. For any Γ_ρ invariant map $h : T^m \to W$, we define a map $h_A : T^m \to L(G)$ by

$$h_A = \sum_{\gamma \in A} \sigma_e(\gamma^{-1}) \circ h \circ \rho(\gamma).$$

Let γ_1 and $\gamma_2 = \gamma_0\gamma_1$ be two elements of Γ belonging to the same right coset of Γ_ρ. Since h is Γ_ρ-invariant and Γ_ρ-action on W is trivial, it is easy to see that

$$\sigma_e(\gamma_2^{-1}) \circ h \circ \rho(\gamma_2) = \sigma_e(\gamma_1^{-1}) \circ h \circ \rho(\gamma_1).$$

Therefore if B is another set containing exactly one element of each coset of Γ_ρ then $h_A = h_B$. Now it is easy to verify that for all γ in Γ,

$$h_A \circ \rho(\gamma) = \sigma_e(\gamma) \circ h_{\gamma A}.$$

Hence h_A is a Γ-equivariant map from (T^m, ρ) to $(L(T^m), \sigma_e)$. We will show that for a suitable choice of h, h_A is nonzero and $h_A(e) = 0$.

Let $\chi_0 \in \widehat{T}^m$ and $x_0 \in T^m$ be as in Lemma 3.1. Define $c_0, c_1, \ldots, c_d \in S^1$ by

$$c_0 = 1, \quad c_i = \chi_0 \circ \rho(\gamma_i)(x_0) \ \forall i = 1, \ldots, d.$$

Then $1, c_1, \ldots, c_d$ are distinct. We choose a continuous map g from S^1 to W such that

$$g(c_d) \neq 0, \quad g(c_i) = 0, \quad i = 0, \ldots, d-1.$$

Since the map $g \circ \chi_0 : T^m \to W$ is Γ_ρ-invariant, from the previous argument it follows that the map $S = (g \circ \chi_0)_A$ is a Γ-equivariant map from (T^m, ρ) to $(L(T)^n, D\sigma)$. Also it is easy to see that S is nonzero and $S(e) = 0$. Now Theorem 2 follows from Lemma 2.1.

The following corollary generalizes earlier results of [AP] and [Wa1].

Corollary 3.1 : *Let A and B be elements of $GL(m, \mathbb{Z})$ and $GL(n, \mathbb{Z})$ respectively. Let k_A be the smallest positive integer i such that A^i has no*

eigenvalue which is a proper root of unity. Then the following two are equivalent.

a) There exists a continuous nonaffine map $f : T^m \to T^n$ satisfying $f \circ A = B \circ f$.

b) 1 is an eigenvalue of B^{k_A}.

Proof : Let Γ be the cyclic group, ρ be the Γ-action on T^m generated by A and σ be the Γ-action on T^n generated by B. Then after suitable identifications we have,

$$\hat{T}^m = \mathbb{Z}^m, \quad F_\rho = \{z \in \mathbb{Z}^m \mid A^i(z) = z \; for\ some\ i\ \}.$$

It is easy to see that A^{k_A} leaves F_ρ invariant. Since no eigenvalue of A^{k_A} is a proper root of unity, it follows that A^{k_A} leaves F_ρ pointwise fixed. Suppose j is another positive integer such that A^j leaves F_ρ pointwise fixed. Then it easy to check that j is a multiple of k_A. Therefore $\Gamma_\rho = k_A\mathbb{Z}$. It is easy to see that the action $\sigma|_{\Gamma_\rho}$ has a nonzero fixed point in $L(\mathbb{R}^n)$ if and only if 1 is an eigenvalue of B^{k_A}. Now the given assertion follows from Theorem 2.

4 Rigidity of factor maps

In this section we will consider the case when Γ is abelian and X_2 is a topological factor of X_1 i.e. there exists a continuous Γ-equivariant map from X_1 onto X_2. We will need the following two results.

Theorem 4.1 (see [Be], Theorem 5.1)**:** *Let Γ be a discrete abelian group, T^n be the n-torus and ρ be an ergodic automorphism action of Γ on T^n. Then there exists an element γ_0 of Γ such that $\rho(\gamma_0)$ is an ergodic automorphism.*

Theorem 4.2 (see [Pa])**:** *Let $X = G/D$ be a nilmanifold and θ be an automorphism of X such that θ induces an ergodic automorphism on the torus $G/[G,G] \circ D$. Then θ is an ergodic automorphism of X.*

If $X = G/D$ is a nilmanifold then by X^0 we denote the torus $G/[G,G]\cdot D$ and by π^0 we denote the projection map from G onto X^0. If ρ is an automorphism action of a discrete group Γ on X then ρ^0 will denote the automorphism action of Γ on X^0 induced by ρ.

We note the following simple consequence of Theorem 4.1 and Theorem 4.2.

Proposition 4.1 : *Let Γ be a discrete abelian group, $X = G/D$ be a nilmanifold and ρ be an automorphism action of Γ on X. Then (X, ρ) is ergodic if and only if (X^0, ρ^0) is ergodic.*

Proof : Let $q : X \to X^0$ denote the projection map. Then it is easy to check that q is a measure preserving Γ-equivariant map from (X, ρ) to (X^0, ρ^0). Therefore ergodicity of (X, ρ) implies ergodicity of (X^0, ρ^0). On the other hand if (X^0, ρ^0) is ergodic then by Theorem 4.1 there exists a γ in Γ such that $\rho^0(\gamma)$ is an ergodic automorphism of X^0. Applying Theorem 4.2 we see that $\rho(\gamma)$ is an ergodic automorphism of X i.e. (X, ρ) is ergodic.

If V is a finite dimensional vector space over a field K then by V^* we denote the dual of V. If Γ is a discrete group and $\rho : \Gamma \to GL(V)$ is an automorphism action of Γ on V then by ρ^* we denote the action of Γ on V^* defined

by

$$\rho^*(\gamma)(q)(v) = q(\rho^*(\gamma)^{-1}v) \quad \forall q \in V^*, v \in V.$$

Proposition 4.2 : *Let Γ be an abelian group and V be a finite dimensional vector space over $\mathbb{R}$. Let $\rho : \Gamma \to GL(V)$ be an automorphism action of Γ on V such that the induced Γ-action on the dual of V has a nontrivial fixed point. Then ρ has nontrivial fixed point in V.*

Proof : By passing to the complexification we see that it is enough to prove the analogous statement when V is a finite dimensional vector space over $\mathbb{C}$. In that case after suitable identifications we can assume that $V = V^* = \mathbb{C}^n$, $\rho : \Gamma \to GL(n, \mathbb{C})$ is a homomorphism and $\rho^* : \Gamma \to GL(n, \mathbb{C})$ is the homomorphism defined by $\rho^*(\gamma) = \rho(\gamma^{-1})^T$. Let us consider the special case when with respect to some basis in $\mathbb{C}^n$ each $\rho(\gamma)$ is given by an upper triangular matrix with equal diagonal entries. In this case it is easy to verify that ρ or $\rho*$ has a nonzero fixed vector in $\mathbb{C}^n$ if and only if for any γ in Γ all the diagonal entries of $\rho(\gamma)$ are equal to 1. To prove the general case we note that since Γ is abelian, there exist subspaces $V_1, V_2, \ldots, V_k$ of $\mathbb{C}^n$ and homomorphisms $\rho_i : \Gamma \to GL(V_i)$; $i = 1, \ldots, k$ such that $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k$, $\rho = \rho_1 \oplus \cdots \oplus \rho_k$ and each ρ_i satisfies the above condition (cf. [Ja], pp. 134).

Proposition 4.3 : *Let σ be an automorphism action of a discrete abelian group Γ on a torus T^n. Then (T^n, σ) is ergodic if and only if there is no nonzero element in $L(T^n)$ with finite σ_e-orbit.*

Proof : Since $T^n = \mathbb{R}^n/\mathbb{Z}^n$, $L(T^n)$ can be identified with $\mathbb{R}^n$. Also σ_e

can be realised as a homomorphism from Γ to $GL(n,\mathbb{Z})$, the dual action σ_e^* can be realised as the homomorphism from Γ to $GL(n,\mathbb{Z})$ which takes γ to $\sigma_e(\gamma^{-1})^T$ and $\widehat{\sigma}$ can be identified with $\sigma_e^*|_{\mathbb{Z}^n}$. Suppose (T^n,σ) is ergodic. Let $\Gamma_0 \subset \Gamma$ be a subgroup of finite index. Then no nonzero element of $\mathbb{Z}^n$ is fixed by Γ_0 under the action σ_e^*. Since $\sigma_e^*(\gamma) \in GL(n,\mathbb{Z})$ for all γ, this implies that no nonzero element of $\mathbb{R}^n$ is fixed by Γ_0 under the action σ_e^*. Applying Proposition 4.1 we see that no nonzero element of $\mathbb{R}^n$ is fixed by Γ_0 under the action σ_e. Now suppose (T^n,σ) is not ergodic. Then there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and a nonzero point z in $\mathbb{Z}^n$ such that z is fixed by Γ_0 under the action σ_e^*. Now from Proposition 4.1 we conclude that there exists a nonzero element in $\mathbb{R}^n$ which is fixed by Γ_0 under the action σ_e.

Proof of Theorem 3 : Suppose (X_2,σ) is not ergodic. By our assumption there exists a continuous Γ-equivariant map f from (X_1,ρ) onto (X_2,σ). If f is nonaffine then there is nothing to prove. Therefore we may assume that there exists a $g_0 \in G_2$ and a continuous homomorphism $\theta : G_1 \to G_2$ such that $f(gD_1) = g_0\theta(g)D_2$ for all g in G_1. Since f is surjective and Γ-equivariant, so is θ. Let θ_0 denote the homomorphism from X_1^0 to X_2^0 induced by θ. Then θ_0 is surjective. Since (X_2,σ) is not ergodic, from Proposition 4.1 it follows that (X_2^0,σ^0) is not ergodic. Let ϕ be an element of X_2^0 such that $\widehat{\sigma^0}$-orbit of ϕ is finite. Since θ^0 is Γ-equivariant, it follows that $\widehat{\rho^0}$-orbit of $\phi \circ \theta^0$ is also finite, which implies that (X_1^0,ρ_0) is not ergodic. Also for any γ in Γ_ρ,

$$\phi \circ \sigma^0(\gamma) \circ \theta^0 = \phi \circ \theta^0 \circ \rho^0(\gamma) = \phi \circ \theta^0.$$

Since θ^0 is a surjective map, this implies that $\phi \circ \sigma^0(\gamma) = \phi$ for all γ in Γ_ρ. Let π_2^0 denote the projection map from G_2 onto X_2^0 and let q denote the map

$\phi \circ \pi_2^0 \circ \exp$. Then $dq : L(G_2) \to \mathbb{R}$ is an element of the dual of $L(G_2)$ such that $dq \circ \sigma_e(\gamma) = dq$ for all γ in Γ_ρ. Now from Proposition 4.2 it follows that there exists a nonzero point in $L(G_2)$ which is fixed by Γ_ρ under the action σ_e. Applying Theorem 2 we see that there exists a continuous nonaffine Γ-equivariant map h from (X_1^0, ρ^0) to (X_2, σ). If π_1^0 denotes the projection map from X_1 to X_1^0 then it is easy to see that $h \circ \pi_1^0$ is a continuous nonaffine Γ-equivariant map h from (X_1, ρ) to (X_2, σ).

Now suppose (X_2, σ) is ergodic. Since by our assumption either $(X_1, \rho) = (X_2, \sigma)$ or X_2 is a torus from Proposition 4.3 it follows that either (X_1, ρ) is ergodic or there is no non-zero element in $L(G_2)$ whose σ_e-orbit is finite. Applying Theorem 1 we conclude that every continuous Γ-equivariant map from (X_1, ρ) to (X_2, σ) is an affine map.

The following examples show that Theorem 3 does not hold if any of the assumptions as in the hypothesis is dropped.

Example 1 : Let Γ be the cyclic group and ρ, σ be the automorphism actions of Γ on $\mathbb{R}/\mathbb{Z}$ generated by the identity automorphism and the automorphism $z \to -z$ respectively. Then it is easy to see that in this case $\Gamma_\rho = \Gamma$ and no nonzero element of $L(\mathbb{R})$ is fixed by Γ_ρ under the action σ_e. Now applying Theorem 2 we conclude that there is no nonaffine continuous Γ-equivariant map from (S^1, ρ) to (S^1, σ). Note that in this case Γ is abelian and neither of the two actions is ergodic.

Example 2 : Fix $n \geq 3$ and define a subgroup Γ of $GL(n, \mathbb{Z})$ by

$$\Gamma = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n-1, \mathbb{Z}), b \in \mathbb{Z}^{n-1} \right\}$$

Let ρ denote the natural action of Γ on $\mathbb{R}^n/\mathbb{Z}^n$. Then it is easy to see that for any $x = (x_1, \ldots, x_n)$ in $L(\mathbb{R}^n)$, the ρ_e-orbit of x is unbounded. Applying Theorem 1 we see that there is no nonaffine continuous Γ-equivariant map from (T^n, ρ) to (T^n, ρ). Note that in this case (T^n, ρ) is not ergodic since the vector $x_0 = (0, \ldots, 0, 1)$ is fixed by the dual action ρ^*.

Example 3 : Suppose $X = G/D$ where G and D are defined by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \ , \ D = \left\{ \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \mid p, q, r \in \mathbb{Z} \right\}$$

Let A be an ergodic automorphism of G/D. If G_0 denotes the center of G then it is easy to see that $G_0/G_0 \cap D$ is isomorphic to S^1. Hence replacing A by A^2 if necessary we may assume that A acts trivially on G_0. Define a nilmanifold X_1 and an automorphism A_1 of X_1 by $X_1 = X \times S^1, A_1 = A \times Id$. Let ρ_1 and ρ denote the automorphism actions of $\mathbb{Z}$ on X_1 and X generated by A_1 and A respectively. Then (X, ρ) is a factor of (X_1, ρ_1). Let $\pi : X_1 \to S^1$ be the the projection map and $h : S^1 \to L(G_0)$ be any nonzero map such that $h(e) = 0$. Then $h \circ \pi$ is a nonzero Γ-equivariant map from (X_1, ρ_1) to $(L(G), \rho_e)$ such that $h \circ \pi(e) = 0$. Applying Lemma 2.1 we see that there exists a nonaffine continuous Γ-equivariant map from (X_1, ρ_1) to (X, ρ).

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